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J. Math. Anal. Appl. 323 (2006) 644–653
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Global existence results for impulsive functional differential equations ☆

Zhiguo Luo *, Jianhua Shen

Department of Mathematics, Hunan Normal University, Changsha 410081, PR China

Received 18 June 2005
Available online 2 December 2005
Submitted by J. Henderson

Abstract

In this paper, we investigate the existence of solutions of impulsive delay differential equation

$$\begin{cases} x'(t) = f(t, x_t), & t \geqslant 0, t \neq \tau_k, k \in N, \\ \Delta x(t) = I_k(x(t)), & t = \tau_k, k \in N. \end{cases}$$

New global existence theorems are established without assuming the global existence of solutions of the corresponding continuous equation.
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Keywords: Global solution; Impulse; Delay differential equation; Lyapunov function

1. Introduction

The theory of impulsive differential equations provides a general framework for mathematical modelling of many real world phenomena. Indeed, differential equations with impulses are a basic tool for studying evolution processes that are subject to abrupt changes in their states (refer to [1,2]). Remarked progress has been made in the theory of impulsive differential equations

☆ This work is supported by the National Nature Science Foundation of PR China (10571050), Hunan Provincial Natural Science Foundation of China (05JJ40013) and Scientific Research Fund of Hunan Provincial Education Department (05C413).
* Corresponding author.
E-mail address: luozg@mail.hunnu.edu.cn (Z. Luo).

0022-247X/$ – see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2005.10.066

in recent years. However, there is only a small amount of work dedicated to the corresponding theory for impulsive delay differential equations.

Let $J \subset R$ be any interval and $D \subset R^n$ be an open set, set $PC(J, D) = \{x : J \to D$: x is continuous everywhere except at the points $t = \tau_k \in J$, and $x(\tau_k^+)$ and $x(\tau_k^-)$ exist in D with $x(\tau_k^-) = x(\tau_k)\}$. For any $x \in PC(J, D)$, the sup-norm $\|x\| = \sup_{s\in J} |x(s)|$, where $|\cdot|$ is a norm in R^n. Then $PC(J, D)$ is a Banach space. Let us consider the impulsive delay differential equation

$$\begin{cases} x'(t) = f(t, x_t), & t \geqslant 0, t \neq \tau_k, k \in N, \\ \Delta x(t) = I_k(x(t)), & t = \tau_k, k \in N, \end{cases} \tag{1.1}$$

where $f : [0, \infty) \times PC([-r, 0], D) \to R^n$, $I_k \in C(D, R^n)$, $k = 1, 2, \ldots,$ $0 < \tau_1 < \tau_2 < \cdots < \tau_k \to \infty$ as $k \to \infty$, $\Delta x(t) = x(t^+) - x(t^-)$, where $x(t^+) = \lim_{h\to 0^+} x(t + h)$ and $x(t^-) = \lim_{h\to 0^+} x(t - h)$ represent the right and left limits of $x(t)$ at t, respectively. For all $t \geqslant 0$, $x_t \in PC([-r, 0], D)$ is defined by $x_t(s) = x(t + s)$ for $-r \leqslant s \leqslant 0$, that is, $x_t(\cdot)$ represents the history of the state from time $t - r$ up to the present time t.

For given $t_0 \geqslant 0$, $\phi \in PC([-r, 0], D)$ and $\omega \in D$, with system (1.1), one associates an initial condition

$$x_{t_0} = \phi, \qquad x(t_0^+) = \omega. \tag{1.2}$$

Definition 1.1. A function $x : [t_0 - r, t_0 + \alpha] \to D$ ($\alpha > 0$) is called a solution of the initial value problem (1.1)–(1.2) if $x(t)$ is continuous for $t \in (t_0, t_0 + \alpha] \backslash \{\tau_k, k = 1, 2, \ldots\}$, $x(\tau_k^+)$ and $x(\tau_k^-)$ exist and $x(\tau_k) = x(\tau_k^-)$, and satisfies (1.1)–(1.2).

Definition 1.2. A function $x : [t_0 - r, t_0 + \beta) \to D$ ($0 < \beta \leqslant \infty$) is said to be a solution of the initial value problem (1.1)–(1.2) if for each $0 < \alpha < \beta$ the restriction of x to $[t_0 - r, t_0 + \alpha]$ is a solution of (1.1)–(1.2).

Subject to the impulsive delay differential equation (1.1), the differential equation

$$y'(t) = f(t, y_t), \quad t \geqslant 0, \tag{1.3}$$

is called the corresponding continuous equation or the corresponding differential equation without impulse effects, of (1.1).

It is well known that the existence problem of global solutions is of fundamental importance for the investigation of qualitative behavior of solutions of differential equations (such as stability and oscillation, etc.). Here, a global solution is one defined on $[0, +\infty)$. In existing literature, the existence of solutions of impulsive functional differential equations was discussed (see, for example, [3–10]). Typically, all known research works on the existence of global solutions to (1.1) take the existence of global solutions of (1.3) with any initial conditions as a basic assumption in order to ensure the existence of global solutions of (1.1). We note that the behavior of solutions of impulsive delay differential equation (1.1) are influenced by the impulses and some of them are different from those of the corresponding continuous equation, i.e., (1.3). Let us consider the following equation:

$$\begin{cases} x'(t) = \frac{2(1+x^2(t-\pi/4))}{(1-x(t-\pi/4))^2}, & t \geqslant 0, t \neq \frac{k\pi}{4}, k \in N, \\ \Delta x(t) = -1, & t = \frac{k\pi}{4}, k \in N, \end{cases} \tag{1.4}$$

whose solution $x(t)$ satisfying $x(t) = \tan t$, $t \in [-\frac{\pi}{4}, 0]$, and $x(0^+) = 0$ is continuable for all $t \geqslant 0$, i.e., it is a global solution. In fact, this solution has the form

$$x(t) = \tan\left(t - \frac{k\pi}{4}\right), \quad t \in \left(\frac{k\pi}{4}, \frac{(k+1)\pi}{4}\right], \ k = 0, 1, 2, \ldots. \tag{1.5}$$

However, the corresponding continuous equation

$$x'(t) = \frac{2(1 + x^2(t - \pi/4))}{(1 - x(t - \pi/4))^2}, \quad t \geqslant 0, \tag{1.6}$$

has the solution $x(t) = \tan t$ satisfying the initial condition $x(t) = \tan t$, $t \in [-\pi/4, 0]$, whose maximal right interval of existence only is $[-\pi/4, \pi/2)$ since $\lim_{t\to\pi/2} x(t) = \infty$.

Therefore, it is important and necessary to establish the sufficient conditions for global existence of solutions for impulsive delay differential equations when the corresponding continuous equation does not enjoy global solutions. To derive our results, we will make use of piecewise continuous Lyapunov like functions and the following fixed point theorem of Schaefer [11] which was discussed and proved also in Smart [12].

Lemma A *(Schaefer). Let $(C, \|\cdot\|)$ be a convex subset of a normal linear space, and let the operator $A : C \to C$ be completely continuous. Define*

$$F(A) = \bigl\{x \in C \colon x = \lambda A x, \ \lambda \in [0, 1]\bigr\}. \tag{1.7}$$

Then either

(i) *the set $F(A)$ is unbounded, or*
(ii) *the operator A has a fixed point in C.*

2. Main results

Firstly, we give an effective sufficient condition to ensure the existence of global solutions of (1.1) by using the Schaefer fixed point theorem.

We introduce the following classes of functions for later use:

$$K_0 = \bigl\{\psi \in C(R^+, R^+) \colon \psi(0) = 0, \ \psi(s) > 0 \text{ for } s > 0\bigr\},$$
$$K = \bigl\{\psi \in K_0 \colon \psi(s) \text{ is strictly increasing}\bigr\},$$
$$KR = \Bigl\{\psi \in K \colon \lim_{s\to\infty} \psi(s) = \infty\Bigr\},$$

and $\nu_0(M) = \{V : R^+ \times S^c(M) \to R^+ \colon V(t, x)$ is continuous on $(\tau_k, \tau_{k+1}] \times S^c(M)$, locally Lipschitzian in x and $V(\tau_k^+, x)$ exists for $k = 1, 2, \ldots\}$, where $M \geqslant 0$ and $S^c(M) = \{x \in R^n \colon |x| \geqslant M\}$.

Definition 2.1. Let $V \in \nu_0(M)$, for any $(t, x) \in R^+ \times S^c(M)^0$, $t \neq \tau_k$, the right hand derivative of $V(t, x)$ with respect to the continuous portion of system (1.1) is defined by

$$D^+V(t, x) = \limsup_{h\to 0^+} \frac{V(t + h, x + hf(t, x)) - V(t, x)}{h},$$

where $S^c(M)^0$ denotes the interior of $S^c(M)$.

To applying the Schaefer fixed point theorem, we will need a compactness criterion for a set $\Lambda \in PC([t_0 - r, T], R^n)$.

Definition 2.2. [13] The set Λ is said to be quasiequicontinuous in $[t_0, T]$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \Lambda$, $k \in N$, $t_1, t_2 \in (\tau_{k-1}, \tau_k] \cap [t_0, T]$, and $|t_1 - t_2| < \delta$, then $|x(t_1) - x(t_2)| < \varepsilon$.

The following lemma gives a necessary and sufficient condition for relative compactness in $PC([t_0 - r, T], R^n)$.

Lemma B *(Compactness criterion [13,14]). The set $\Lambda \subset PC([t_0 - r, T], R^n)$ is relatively compact if and only if*

(1) *Λ is uniformly bounded, that is, $\|x\|_{PC} \leqslant B$ for each $x \in \Lambda$ and some $B > 0$;*
(2) *Λ is quasiequicontinuous in $[t_0 - r, T]$.*

We introduce the following conditions:

(H2.1) $f(t, \psi)$ is continuous on $(t, \psi) \in R^+ \times PC([-r, 0], R^n)$, and $I_k \in C(R^n, R^n)$, $k = 1, 2, \ldots$.
(H2.2) There exist constants M, H such that $|x + I_k(x)| \leqslant H$ for $k = 1, 2, \ldots$, whenever $|x| \leqslant M$.

We assume that $t_0 \in [\tau_l, \tau_{l+1})$ for some integer $l \geqslant 0$ and $t_0 < \tau_{l+1} < \tau_{l+2} < \cdots < \tau_{l+m} < T$. A straightforward calculation shows that the existence of solution $x(t, t_0, \phi, \omega)$ of (1.1) is equivalent to the existence of solution of the corresponding impulsive integral equation

$$x(t) = \begin{cases} \phi(t - t_0), & t \in [t_0 - r, t_0], \\ \omega + \int_{t_0}^{t} f(s, x_s)\, ds + \sum_{t_0 < \tau_k < t} I_k(x(\tau_k)), & t \in (t_0, T]. \end{cases} \tag{2.1}$$

To apply Schaefer theorem, let $0 \leqslant \lambda \leqslant 1$ and consider

$$\begin{cases} x' = \lambda f(t, x_t), & t \neq \tau_k,\ k = l+1, l+2, \ldots, l+m, \\ \Delta x = \lambda I_k(x), & t = \tau_k,\ k = l+1, l+2, \ldots, l+m, \\ x(t) = \lambda\phi(t - t_0),\ x(t_0^+) = \lambda\omega, & t \in [t_0 - r, t_0], \end{cases} \tag{2.2}$$

or the equivalent integer equation

$$x(t) = \begin{cases} \lambda\phi(t - t_0), & t \in [t_0 - r, t_0], \\ \lambda\left[\omega + \int_{t_0}^{t} f(s, x_s)\, ds + \sum_{t_0 < \tau_k < t} I_k(x(\tau_k))\right], & t \in (t_0, T]. \end{cases} \tag{2.3}$$

Theorem 2.1. *Assume that there exist $p \in PC(R^+, R^+)$, $a, b, \psi \in KR$, c, ψ_k $(k = 1, 2, \ldots) \in K_0$ and $V \in \nu_0(M)$ for some $M > 0$ such that*:

(i) *$b(|x|) \leqslant V(t, x) \leqslant a(|x|)$, $(t, x) \in R^+ \times S^c(M)$;*
(ii) *$\psi_k(s) \leqslant \psi(s) < s$ for $s > 0$ and $V(\tau_k^+, x + I_k(x)) \leqslant \psi_k(V(\tau_k, x))$ for x, $x + I_k(x) \in S^c(M)$, $k = 1, 2, \ldots$;*

(iii) *for any solution $x(t)$ of* (2.2), $V(t,x(t)) \geqslant \psi(V(t+s,x(t+s)))$ $(s \in [-r,0])$ *implies that*

$$D^+V(t,x) \leqslant p(t)c\big(V(t,x)\big), \quad (t,x) \in R^+ \times S^c(M)^0,\ t \neq \tau_k; \tag{2.4}$$

(iv) *there exist* λ_1, λ_2 *such that*

$$\lambda_2 > a\big(b^{-1}\big(\psi^{-1}\big(a(h)\big)\big)\big), \quad \textit{where } h = \max\big\{b^{-1}(\lambda_1), M, H\big\}, \tag{2.5}$$

and

$$\int_{\tau_k}^{\tau_{k+1}} p(s)\,ds + \int_{y}^{\psi(y)} \frac{ds}{c(s)} < 0, \quad \lambda_1 \leqslant y \leqslant \lambda_2,\ k = 1, 2, \ldots. \tag{2.6}$$

Then there exists constant $\alpha > h$ such that for any $t_0 \geqslant 0$ and $(\phi,\omega) \in PC([-r,0],R^n) \times R^n$, $\|\phi\| < \alpha$, the solution $x = x(t,t_0,\phi,\omega)$ of (1.1) *exists on* $[t_0 - r, \infty)$.

Proof. Let $T > t_0$ be given. We will prove that there is a solution $x(t,t_0,\phi,\omega)$ of (1.1) on $[t_0 - r, T]$. Set $I = [t_0 - r, T]$, $J = [t_0, T]$ and consider the Banach space $PC = PC(I, R^n)$. For any $x \in PC$, define the integral operator A by

$$[Ax](t) = \begin{cases} \phi(t - t_0), & t \in [t_0 - r, t_0], \\ \omega + \int_{t_0}^{t} f(s,x_s)\,ds + \sum_{t_0 < \tau_k < t} I_k(x(\tau_k)), & t \in (t_0, T]. \end{cases} \tag{2.7}$$

It is easy to see that A maps PC into PC.

The conditions of Schaefer theorem will be verified by the following three steps.

Step 1. A is continuous. Let $\{x_n\}$ be a sequence such that $x_n \to x$ as $n \to \infty$ in PC. Then there is an constant $B_1 > 0$ such that $\|x_n\|_{PC} \leqslant B_1$ for all $n \in N$ and $\|x\|_{PC} \leqslant B_1$. In view of the continuity of $f(t,x)$ and $I_k(x)$ on $|x| \leqslant B_1$ and $t_0 - r \leqslant t \leqslant T$, we have

$$\|Ax_n - Ax\|_{PC} \leqslant \int_{t_0}^{T} \big|f(s,x_{ns}) - f(s,x_s)\big|\,ds + \sum_{k=l+1}^{l+m} \big|I_k\big(x_n(\tau_k)\big) - I_k\big(x(\tau_k)\big)\big| \to 0.$$

Therefore A is continuous.

Step 2. A maps bounded sets into bounded sets. Indeed, for given $B_1 > 0$, if $x \in PC$ and $\|x\|_{PC} \leqslant B_1$, then $\|x_s\| \leqslant B_1$ and so there is a $B_2 > 0$ with $|f(t,x_t)| \leqslant B_2$ and $|I_k(x(t))| \leqslant B_2$ for $t_0 \leqslant t \leqslant T$ and $k = l+1, l+2, \ldots, l+m$. We have for $t \in J$

$$\big|[Ax](t)\big| \leqslant |\omega| + \int_{t_0}^{t} \big|f(s,x_s)\big|\,ds + \sum_{t_0 < t_k < t} \big|I_k\big(x(\tau_k)\big)\big|.$$

Thus

$$\|Ax\|_{PC} \leqslant |\omega| + (T - t_0 + m)B_2 := B.$$

$A(PC)$ is uniformly bounded.

Step 3. A maps bounded sets into quasiequicontinuous sets. Indeed, for any $t_1, t_2 \in (\tau_{k-1}, \tau_k] \cap [t_0, T]$, $t_2 > t_1$, $x \in PC$, we have

$$\begin{aligned}\bigl|[Ax](t_2) - [Ax](t_1)\bigr| &\leqslant \int_{t_1}^{t_2} \bigl|f(s, x_s)\bigr|\,ds + \sum_{t_1 < \tau_k \leqslant t_2} \bigl|I_k\bigl(x(\tau_k)\bigr)\bigr| \\ &\leqslant B_2|t_2 - t_1| + \sum_{t_1 < \tau_k \leqslant t_2} B_2.\end{aligned}$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero. The quasiequicontinuity for the cases $t_1 < t_2 \leqslant t_0$ and $t_1 < t_0 \leqslant t_2$ is obvious.

As a consequence of Steps 1–3, A is completely continuous. To complete the proof of the theorem, it suffices to prove the following step.

Step 4. There exists a constant $B > 0$ such that for any $x \in \Lambda := \{x \in PC\colon x = \lambda Ax,\ 0 \leqslant \lambda \leqslant 1\}$, $\|x\|_{PC} < B$. From (2.6) we see that there exist positive numbers $\alpha > h = \max\{b^{-1}(\lambda_1), M, H\}$ such that $\lambda_2 > a(b^{-1}(\psi^{-1}(a(\alpha))))$. Set $B = b^{-1}(\psi^{-1}(a(\alpha)))$. Then $\lambda_2 > a(B)$. Note that $\alpha > M$ and $b(B) = \psi^{-1}(a(\alpha)) \geqslant a(\alpha) \geqslant b(\alpha)$, thus $B \geqslant \alpha$ and $\lambda_2 > a(B) \geqslant a(\alpha) \geqslant b(\alpha) > \lambda_1$. Let $x \in \Lambda$ with $\|\phi\| < \alpha$, $|\omega| < \alpha$, and set $m(t) = V(t, x(t))$. We will prove that $\|x\|_{PC} < B$ by the following four claims:

Claim 1. *$|x(t)| < \alpha \leqslant B$ and so $m(t) \leqslant a(|x(t)|) < a(\alpha) = \psi(b(B)) < b(B)$ for $t_0 - r \leqslant t \leqslant t_0$. In fact, for $t \in [t_0 - r, t_0]$, $|x(t)| = |\phi(t - t_0)| \leqslant \|\phi\| < \alpha$.*

Claim 2. *$|x(t)| < B$ for $t_0 \leqslant t \leqslant \tau_{l+1}$. Otherwise, $|x(t)| \geqslant B$ for some $t \in [t_0, \tau_{l+1}]$. Set $t^* = \inf\{t \in [t_0, \tau_{l+1}]\colon |x(t)| \geqslant B\}$. Then $t^* \in (t_0, \tau_{l+1}]$, $|x(t^*)| = B$, $m(t^*) \geqslant b(|x(t^*)|) = b(B)$ and $|x(t)| < B$ for $t \in [t_0, t^*)$. Therefore, set $\tilde{t} = \inf\{t \in [t_0, t^*]\colon m(t) \geqslant b(B)\}$, $\tilde{t} \in (t_0, t^*]$, $m(\tilde{t}) = b(B) > \psi(b(B))$ and $m(t) < b(B)$, $t_0 - r \leqslant t < \tilde{t}$.*

Define $\hat{t} = \sup\{t \in (t_0, \tilde{t}]\colon m(t) < \psi(b(B))\}$. Clearly, $\hat{t} \in [t_0, \tilde{t})$ and $m(t) \geqslant \psi(b(B))$ for $t \in (\hat{t}, \tilde{t}]$. We have $m(t) \geqslant \psi(b(B)) > \psi(m(t+s))$ for $t \in (\hat{t}, \tilde{t}]$, $s \in [-r, 0]$. Moreover, since $a(\alpha) = \psi(b(B)) \leqslant m(t) \leqslant a(|x(t)|)$, $t \in (\hat{t}, \tilde{t}]$, $|x(t)| \geqslant \alpha > M$, $t \in (\hat{t}, \tilde{t}]$. By condition (2.4) we see that

$$D^+ m(t) \leqslant p(t) c\bigl(m(t)\bigr), \quad t \in (\hat{t}, \tilde{t}]. \tag{2.8}$$

Moreover, $m(t) \geqslant \psi(b(B)) = a(\alpha) > \lambda_1$ and $m(t) \leqslant a(|x(t)|) < a(B) < \lambda_2$ for $t \in (\hat{t}, \tilde{t}]$. Since $c \in K_0$, we see that $c(m(t)) > 0$ for $t \in (\hat{t}, \tilde{t}]$. Thus for any $t_1, t_2 \in [\hat{t}, \tilde{t}]$ with $t_1 < t_2$, from (2.8) we have

$$\int_{m(t_1^+)}^{m(t_2)} \frac{ds}{c(s)} \leqslant \int_{t_1}^{t_2} p(s)\,ds. \tag{2.9}$$

It is easy to show that $m(\hat{t}^+) \leqslant \psi(b(B))$. In fact, if $\hat{t} = t_0$, then since $|x(t_0^+)| = |\omega| < \alpha$, $m(\hat{t}^+) = m(t_0^+) < a(\alpha) = \psi(b(B))$. If $\hat{t} > t_0$, then $m(\hat{t}^+) = \psi(b(B))$. Moreover, $\lambda_1 < b(\alpha) < b(B) \leqslant a(B) < \lambda_2$.

Thus, we get by (2.5) and (2.9)

$$0 > \int_{\tau_l}^{\tau_{l+1}} p(s)\,ds + \int_{b(B)}^{\psi(b(B))} \frac{ds}{c(s)} \geqslant \int_{\hat{t}}^{\tilde{t}} p(s)\,ds + \int_{b(B)}^{\psi(b(B))} \frac{ds}{c(s)}$$

$$\geqslant \int_{m(\hat{t}^+)}^{m(\tilde{t})} \frac{ds}{c(s)} + \int_{b(B)}^{\psi(b(B))} \frac{ds}{c(s)} \geqslant \int_{\psi(b(B))}^{b(B)} \frac{ds}{c(s)} + \int_{b(B)}^{\psi(b(B))} \frac{ds}{c(s)} = 0,$$

which is a contradiction and so $|x(t)| < B$ for $t_0 \leqslant t \leqslant \tau_{l+1}$.

Similarly, we can prove that $m(t) < b(B) = \psi^{-1}(a(\alpha))$ for $t_0 \leqslant t \leqslant \tau_{l+1}$.

Claim 3. $|x(\tau_{l+1}^+)| < B$. *If* $|x(\tau_{l+1}^+)| \geqslant B$, *then we have two possible cases*:

Case 1. $|x(\tau_{l+1})| \leqslant M$. *Then by (H2.2)* $|x(\tau_{l+1}^+)| = |x(\tau_{l+1}) + I_{l+1}(x(\tau_{l+1}))| \leqslant H < \alpha \leqslant B \leqslant |x(\tau_{l+1}^+)|$, *which is a contradiction.*

Case 2. $|x(\tau_{l+1})| > M$. *Since* $m(\tau_{l+1}) < b(B)$, $|x(\tau_{l+1}^+)| \geqslant B > M$, *then* $b(B) \leqslant b(|x(\tau_{l+1}^+)|) \leqslant V(\tau_{l+1}^+, x(\tau_{l+1}^+)) = V(\tau_{l+1}^+, x(\tau_{l+1}) + I_{l+1}x(\tau_{l+1})) \leqslant \psi_{l+1}(V(\tau_{l+1}, x(\tau_{l+1}))) < a(\alpha) \leqslant b(B)$, *we get a contradiction.*

Claim 4. *Similar to Claims* 2 *and* 3 *one can prove that* $|x(t)| < B$, $m(t) < b(B) = \psi^{-1}(a(\alpha))$ *for* $\tau_{l+1} < t \leqslant \tau_{l+2}$ *and* $|x(\tau_{l+2}^+)| < B$. *By a simple induction we can prove in general that* $|x(t)| < B$, $m(t) < b(B) = \psi^{-1}(a(\alpha))$ *for* $\tau_{l+i} < t \leqslant \tau_{l+i+1}$, $i = 1, 2, \ldots, m-1$, *and* $|x(t)| < B$ *for* $\tau_{l+m} < t \leqslant T$.

Now all conditions of Schaefer theorem are satisfied and therefore the proof is complete. □

In the following, we establish a general global existence result for (1.1) by checking when a solution of (1.3) with any initial condition $y_{t_0} = \phi$, $y(t_0^+) = \omega$ can arrive at nearest one impulse time. The example (1.4) then can be illustrated using this result.

Theorem 2.2. *Assume the following conditions are satisfied*:

(i) *For any* $t_0 \geqslant 0$ *and* $(\phi, \omega) \in PC([-r, 0], D) \times D$, *there exists one solution* $y(t) = y(t, t_0, \phi, \omega)$ *of* (1.3) *whose maximal right interval of existence is* $[t_0, t_0 + a)$, *and* $\inf\{a\colon (t_0, \phi, \omega) \in [0, \infty) \times PC([-r, 0], D) \times D\} = \delta > 0$.
(ii) $0 < \tau_k - \tau_{k-1} < \delta$ *for all* $k = 1, 2, \ldots.$
(iii) $x \in D$ *implies that* $x + I_k(x) \in D$ *for all* $k = 1, 2, \ldots.$

Then there exists one solution $x(t) = x(t, t_0, \phi, \omega)$ *of* (1.1) *defined on* $[t_0 - r, \infty)$.

Proof. Let $t_0 \in [\tau_{m-1}, \tau_m)$ for some $m \in N$, and let $y_0(t)$ be the solution of the equation

$$\begin{cases} y'(t) = f(t, y_t), & t \geqslant t_0, \\ y_{t_0} = \phi, & y(t_0^+) = \omega, \end{cases}$$

defined on $[t_0 - r, t_0 + a_0)$ $(a_0 > 0)$. By condition (ii), $\tau_m \in (t_0, t_0 + a_0)$. Thus, $(y_0)_{\tau_m} \in PC([-r, 0], D)$ and $y_0(\tau_m) \in D$. Using condition (iii), we have $y_0(\tau_m) + I_m(y_0(\tau_m)) \in D$. Then we may let $y_1(t)$ be the solution of the equation

$$\begin{cases} y'(t) = f(t, y_t), & t \geqslant \tau_m, \\ y_{\tau_m} = (y_0)_{\tau_m}, & y(\tau_m^+) = y_0(\tau_m) + I_m(y_0(\tau_m)), \end{cases}$$

defined on $[\tau_m, \tau_m + a_1)$ $(a_1 > 0)$. It is clear that $\tau_{m+1} \in (\tau_m, \tau_m + a_1)$. Thus, $(y_1)_{\tau_{m+1}} \in PC([-r, 0], D)$ and $y_1(\tau_{m+1}) \in D$, and so $y_1(\tau_{m+1}) + I_{m+1}(y_1(\tau_{m+1})) \in D$. Then, we may again formulate an equation of the form

$$\begin{cases} y'(t) = f(t, y_t), & t \geqslant \tau_{m+1}, \\ y_{\tau_{m+1}} = (y_1)_{\tau_{m+1}}, & y(\tau_{m+1}^+) = y_1(\tau_{m+1}) + I_{m+1}(y_1(\tau_{m+1})), \end{cases}$$

whose solution $y_2(t)$ exists on the interval $[\tau_{m+1}, \tau_{m+1} + a_2)$ $(a_2 > 0)$. Repeating this procedure, by induction, we may obtain the solution $y_k(t)$ of the equation

$$\begin{cases} y'(t) = f(t, y_t), & t \geqslant \tau_{m+k-1}, \\ y_{\tau_{m+k-1}} = (y_{k-1})_{\tau_{m+k-1}}, & y(\tau_{m+k-1}^+) = y_{k-1}(\tau_{m+k-1}) + I_{m+k-1}(y_{k-1}(\tau_{m+k-1})), \end{cases}$$

defined on $[\tau_{m+k-1}, \tau_{m+k-1} + a_k)$ $(a_k > 0)$, $k = 1, 2, \ldots$. Finally, we define

$$x(t) = \begin{cases} y_0(t), & t \in [t_0 - r, \tau_m], \\ y_1(t), & t \in (\tau_m, \tau_{m+1}], \\ \cdots \\ y_k(t), & t \in (t_{m+k-1}, \tau_{m+k}],\ k = 1, 2, \ldots. \end{cases}$$

Then it is easy to verify that $x(t)$ is the solution of (1.1) defined on $[t_0 - r, \infty)$ such that $x_{t_0} = \phi$, $x(t_0^+) = \omega$. The proof is complete. □

Example. Consider the impulsive delay differential equation

$$x'(t) = \left(1 + \frac{1}{25}x^2\left(t - \frac{\pi}{4}\right)\right)\sin\bigl(e^{x(t)} + t\bigr), \quad t \neq \frac{k}{2},\ k = 1, 2, \ldots, \tag{2.10}$$

$$\Delta x(t) = -\frac{4}{5}x(t), \quad t = \frac{k}{2},\ k = 1, 2, \ldots. \tag{2.11}$$

Then (2.10)–(2.11) satisfies all conditions of Theorem 2.1 for $t_0 = 0$, $\phi(t) = \tan t$ $(t \in [-\pi/4, 0])$, $x(0^+) = 0$.

In fact, let $V(t, x) = V(x) = |x|$, $M = H = \lambda_1 = 1$, $a(s) = b(s) = s$, $p(t) \equiv 1$, $c(s) = 1 + s^2$, $\psi(s) = \psi_k(s) = s/5$, $\alpha = 1 + \varepsilon$, $\lambda_2 = 5 + 2\varepsilon$ for $\varepsilon > 0$ sufficiently small. Then for any solution $x(t)$ of (2.10)–(2.11) such that

$$V\bigl(t, x(t)\bigr) \geqslant \psi\bigl(V\bigl(t + s, x(t + s)\bigr)\bigr) \quad \text{for } s \in [-\pi/4, 0],$$

we have

$$D^+V\bigl(t, x(t)\bigr) \leqslant \bigl(1 + x^2(t)\bigr), \quad \bigl(t, x(t)\bigr) \in R^+ \times S^c(M)^0,\ t \neq k/2,$$

and

$$\int_{\tau_k}^{\tau_{k+1}} p(s)\,ds + \int_y^{\psi(y)} \frac{ds}{c(s)} = \int_{k/2}^{(k+1)/2} ds + \int_y^{y/5} \frac{ds}{1 + s^2} = \frac{1}{2} + \arctan\frac{y}{5} - \arctan y.$$

Set

$$u(y) = \frac{1}{2} + \arctan\frac{y}{5} - \arctan y,$$

then

$$u'(y) = \frac{4(y^2 - 5)}{(25 + y^2)(1 + y^2)}.$$

Hence, when $1 \leqslant y \leqslant \sqrt{5}$, $u'(y) \leqslant 0$,

$$u(y) \leqslant u(1) = \frac{1}{2} + \arctan\frac{1}{5} - \arctan 1 \approx -0.088 < 0,$$

and when $\sqrt{5} \leqslant y$, $u'(y) \geqslant 0$. Since

$$u(5) = \frac{1}{2} + \arctan 1 - \arctan 5 \approx -0.088 < 0,$$

so when $\sqrt{5} \leqslant y \leqslant 5 + 2\varepsilon$ ($\varepsilon > 0$ sufficiently small),

$$u(y) \leqslant u(5 + 2\varepsilon) < 0.$$

Thus, when $1 \leqslant y \leqslant 5 + 2\varepsilon$, we have

$$\int_{\tau_k}^{\tau_{k+1}} p(s)ds + \int_{y}^{\psi(y)} \frac{ds}{c(s)} < 0.$$

Thus, the solution of (2.10)–(2.11) satisfying $t_0 = 0$, $\phi(t) = \tan t$ ($t \in [-\pi/4, 0]$), $x(0^+) = 0$ is continuable for all $t \geqslant 0$.

Acknowledgment

The authors express their sincere thanks to the referee for valuable suggestions.

References

[1] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
[2] D.D. Bainov, P.S. Simenov, Systems with Impulse Effect: Stability Theory and Applications, Ellis Horwood, Chichester, 1989.
[3] S.V. Krishna, A.V. Anokhin, Delay differential systems with discontinuous initial data and existence and uniqueness theorems for systems with impulses and delay, J. Appl. Math. Stoch. Anal. 7 (1) (1994) 49–67.
[4] J. Shen, Global existence and uniqueness, oscillation and nonoscillation of impulsive delay differential equations, Acta Math. Sinica 40 (1) (1997) 53–59.
[5] G. Ballinger, X. Liu, Existence and uniqueness results for impulsive delay differential equations, Dyn. Contin. Discrete Impuls. Syst. 5 (1999) 579–591.
[6] D. Cheng, J. Yan, Global existence and asymptotic behavior of solutions of second order nonlinear impulsive differential equations, Int. J. Math. Math. Sci. 25 (2001) 175–182.
[7] G. Marino, P. Pielramala, L. Muglia, Impulsive neutral integrodifferential equations on unbounded intervals, Mediterr. J. Math. 1 (2004) 3–42.
[8] M. Benchohra, J. Henderson, S.K. Ntouyas, An existence result for first order impulsive functional differential equations in Banach spaces, Comput. Math. Appl. 42 (10–11) (2001) 1303–1310.
[9] J.R. Graef, A. Ouahab, Some existence results and uniqueness solutions for functional impulsive differential equations with variable times in Fréchet spaces, Dyn. Contin. Discrete Impuls. Syst., in press.

[10] X. Liu, L. Chen, Global dynamics of the periodic logistic system with periodic impulsive perturbations, J. Math. Anal. Appl. 289 (2004) 279–291.
[11] H. Schaefer, Über die Methode der a priori Schranken, Math. Ann. 129 (1955) 415–416.
[12] D.R. Smart, Fixed Point Theorem, Cambridge Univ. Press, Cambridge, 1980.
[13] J.K. Hale, Theory of Functional Differential Equations, Springer, New York, 1977.
[14] P.S. Simenov, D.D. Bainov, On the asymptotic equivalence of systems with impulse effect, J. Math. Anal. Appl. 135 (1988) 591–610.